

Driving linear systems to non-classical states with the help of noise

Marco G. Genoni,¹ Stefano Mancini,² and Alessio Serafini³

¹*QOLS, Blackett Laboratory, Imperial College London, London SW7 2BW, UK**

²*School of Science and Technology, University of Camerino, I-62032 Camerino, Italy
and INFN, Sezione di Perugia, I-06123 Perugia, Italy*

³*Department of Physics & Astronomy, University College London, Gower Street, London WC1E 6BT, United Kingdom*

We study the possibility of taking bosonic systems subject to quadratic Hamiltonians and a noisy thermal environment to non-classical stationary states by weak Gaussian measurements and conditioned linear driving. We derive general analytical upper bounds for the single mode squeezing and multimode entanglement at steady state, depending only on the Hamiltonian parameters and on the number of thermal excitations of the bath. Our findings show that, rather surprisingly, larger number of thermal excitations in the bath allow for larger steady-state squeezing and entanglement if the efficiency of the Gaussian measurements conditioning the feedback loop is high enough. Such efficiencies are included in our exact treatment, which allows us to determine efficiency thresholds for the noise-enhancement of quantum resources to take place.

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All quantum technologies hinge on establishing controlled interactions between different constituents of quantum systems whilst reducing unwanted interactions with an environment, which give rise to decoherence. In dealing with environmental decoherence, two main paradigms have emerged over the last fifteen years: one may either attempt to decouple the relevant, logical degrees of freedom from the environment by various techniques (*e.g.*, decoherence free subspaces [1], error correction [2], dynamical decoupling [3]), and then proceed to process the quantum information coherently (*e.g.*, in gate-based models of quantum computation, through unitary operations), or one may try to manipulate the noisy, non-unitary evolution of the system directly, tailoring it to suit one's aims.

The second viewpoint, which one might broadly refer to as the ‘dissipative’ approach to quantum information processing, has a long tradition, going back to the first proposals for reservoir engineering [4], and has recently been compounded by the design of a model for dissipative, non-unitary quantum computation [5]. It has hence been repeatedly shown, in various contexts and settings, that working with the environment rather than against it may lead to forms of cooperation whereby the environment contributes to enhance certain coherent tasks performed on the system, often in a rather counterintuitive manner [6–25]. This Letter is the account of such cooperation in the broad setting of controlled dissipative dynamics in linear Gaussian systems [1, 27–33]. We shall consider a system of n bosonic modes subject to a quadratic Hamiltonian and to dissipation in a thermal environment with average excitation number N , and show that the maximal squeezing and entanglement achievable by continuous linear feedback control grows with N , that is with the temperature of the bath. We will apply our results to various quadratic Hamiltonians and also study quantitatively the role played by the efficiency of the weak measurements that condition the feedback loop.

Notation and background. – We consider a system of n bosonic modes described by the vector of canonical operators $\hat{\mathbf{R}} = (\hat{x}_1, \hat{p}_1, \dots, \hat{x}_n, \hat{p}_n)^\top$, with commutation relations

encoded by the anti-symmetric symplectic form Ω , as per $[\hat{R}_j, \hat{R}_k] = i\Omega_{jk}$ ($\hbar = 1$ throughout the paper).

Being comprised of Gaussian noise, Gaussian averages, and Hamiltonian evolutions of the first (‘linear driving’) and second (‘canonical’, or ‘symplectic’) order in the canonical operators, our dynamics will only involve Gaussian states, which are entirely described by first and second statistical moments of the canonical operators [34]. The second moments of a Gaussian state ϱ , in particular, will be represented by a $2n \times 2n$ ‘covariance matrix’ (CM) σ : $\sigma_{jk} = \text{Tr}(\{\hat{R}_j, \hat{R}_k\}\varrho) - 2\text{Tr}(\hat{R}_j\varrho)\text{Tr}(\hat{R}_k\varrho)$, which satisfies the well known Robertson-Schrödinger uncertainty relation:

$$\sigma + i\Omega \geq 0. \quad (1)$$

This is a necessary and sufficient condition for a CM to represent a physical Gaussian state ϱ [35].

Let us begin by considering the most general time-independent quadratic Hamiltonian acting on the system:

$$\hat{H} = (1/2)\hat{\mathbf{R}}^\top H \hat{\mathbf{R}},$$

where the ‘Hamiltonian matrix’ H is a generic symmetric matrix. We will later on modify the Hamiltonian to include a time-dependent linear term which will exert the feedback action on the system. Besides, the system is linearly coupled to a large uncorrelated Markovian bath, whose action can be described, in the standard input-output formalism [1, 36], by a stochastic Langevin equation for the canonical operators $\hat{\mathbf{R}}$:

$$\frac{d\hat{\mathbf{R}}}{dt} = A\hat{\mathbf{R}} + \sqrt{\kappa}\hat{\mathbf{R}}_{in},$$

where A is the ‘drift matrix’, given by $A = (\Omega H - \kappa \mathbb{1}_{2n})/2$ ($\mathbb{1}_d$ standing for the d -dimensional identity matrix), κ is the loss rate of the system and $\hat{\mathbf{R}}_{in}$ is the time-dependent vector of the bath’s operators coupled to all the different system’s modes [1]. We will consider the most general case of Gaussian white noise, with bath’s correlation functions encoded in the diffusion matrix D according to $\kappa\langle\hat{R}_{in,j}(t)\hat{R}_{in,k}(t')\rangle =$

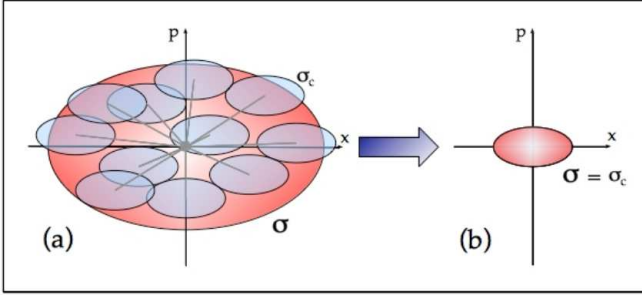


FIG. 1. Heuristic phase-space representation of an optimal linear feedback action. The unconditional state is a Gaussian average, with CM σ , of conditional Gaussian states with the same CM σ_c and different centres in phase space (a). The optimal Markovian choice for the linear driving term, represented by gray arrows in (a) and by $u(t)$ in the Hamiltonian, cancels the first moments of the conditional state, thus making it coincide with the unconditional averaged one (b).

$D_{jk}\delta(t-t')$. Finite temperatures of the environment are incorporated in our treatment by choosing $D = \bigoplus_{j=1}^M (1 + 2N_j) \mathbb{1}_2$, where N_j represents the number of the thermal excitations in the bath of mode j [37]. The free (in the sense that no monitoring or feedback actions have been introduced yet) dynamics of the second moments under such conditions is described by $d\sigma/dt = A\sigma + \sigma A^T + D$. If the system is stable, in the sense of admitting a steady state, it must be $(A + A^T) < 0$, which we will assume in what follows. Also, from now on we will set $\kappa = 1$, and use the loss rate as a unit of frequency. Notice that, to simplify our notation, we have assumed the same loss rate κ affecting all modes. This condition could be relaxed and, in fact, all our general results will be presented in terms of generic drift and diffusion matrices, which allow for generic loss rates.

Let us now define the output field $\hat{\mathbf{R}}_{out}$, by the boundary condition $\hat{\mathbf{R}}_{out} = \sqrt{\kappa}\hat{\mathbf{R}} - \hat{\mathbf{R}}_{in}$ [1]. Monitoring the output field $\hat{\mathbf{R}}_{out}$, which is effectively coupled to the system's field, corresponds to performing a weak measurement on the latter: we will now include in our dynamics the most general Gaussian “general-dyne detection” on the field $\hat{\mathbf{R}}_{out}$ [1], encompassing all homodyne detections, both direct and resorting to ancillary modes (and hence heterodyne detection too). Such measurements are the most general allowing for a continuous, though stochastically fluctuating, monitored evolution of the system if one assumes, as customary in the framework of ‘continuous’ feedback control, that the monitoring of the environment happens on time-scales which are much shorter than the typical system's response time. The detailed description of an arbitrary general-dyne measurement and of the associated monitored dynamics is rather cumbersome. Fortunately, however, our results can be presented by introducing only minimal technicalities (see the supplemental material for a systematic treatment). It will suffice here to say that the effect of any general-dyne measurement of $\hat{\mathbf{R}}_{out}$ can be described in terms of an unravelling matrix U . The knowledge of U allows one

to reconstruct the specific Gaussian measurement by which the environment is being monitored. Furthermore, we allow for a dependence of U on the parameter η , quantifying the overall efficiency of the measurements (in the sense that, before any measurement, a proportion $(1 - \eta)$ of the signal, with $0 \leq \eta \leq 1$, is lost). We will hence be able to include efficiencies in our analytic treatment.

The continuous monitoring of the output field is recorded in the current $y(t)$, and leads to a conditional dynamics which does not alter the Gaussian character of the system's state (here, “conditional” refers to the conditioning due to the knowledge of the weak measurements' outcomes). In the following, we shall distinguish between the conditional state of the system ϱ_c , with CM σ_c , and the time-averaged, ‘unconditional’ state $\varrho = (1/\Delta t) \int_t^{t+\Delta t} \varrho_c(s) ds$, with CM σ , where Δt is an integration interval much larger than the typical time-scale of the stochastic fluctuations of the measured current [38]. As we mentioned, the dynamics of the conditional state ϱ_c is Gaussian, with stochastic fluctuations (depending on the measured current) affecting the first moments, but an entirely *deterministic* evolution for the matrix of second moments σ_c . This fact, as we will see, is essential to our discussion. In fact, the white-noise fluctuations of the first moments are so fast that one is left with the average, unconditional evolution of the quantum state to all practical purposes. But, as depicted in Fig. 1, the unconditional state is just a Gaussian state resulting from the average of conditional Gaussian states with the same CM σ_c and different first moments (centres of their positions in phase space). It is very easy to see that, under such an average, all the figures of merit we are going to consider can only decrease. Hence, for given general-dyne measurement, *the best case scenario for any of our figures of merit would be one where the fluctuations of the first moments cancel out and the average unconditional state coincides with the conditional state*. It turns out that such a situation can always be arranged by adding a linear Markovian feedback action to the Hamiltonian:

$$\hat{H}_f = -\hat{\mathbf{R}}^T \Omega B y(t), \quad (2)$$

where $y(t)$ is the general-dyne current and B is a matrix completely determined by the unravelling matrix U (see the supplemental material for the explicit expression of B). Markovian feedback is therefore always optimal to our aims and we will hence restrict to it in the following. Before proceeding, let us briefly mention that the dynamics of the averaged, unconditional second moments under a linear Markovian feedback action like that of Eq. (13) can still be treated analytically and is of the form $d\sigma/dt = A'\sigma + \sigma A'^T + D'$ (the modified drift and diffusion matrices are given in the supplemental material).

In view of the above, in order to optimise the steady state squeezing or entanglement, one has just to optimise the relevant figure of merit for the conditional state ϱ_c , and then apply the Markovian feedback strategy that ensures $\varrho = \varrho_c$ (see Fig. 1). The optimization over the set of conditional states does not need to go into the details of the conditional dynam-

ics but can instead be tackled by resorting to a general mathematical result: given drift matrix A and diffusion matrix D , a CM σ_c is a stabilising solution of the deterministic conditional dynamics of the second moments if and only if [29]

$$A\sigma_c + \sigma_c A^\top + D \geq 0. \quad (3)$$

General results. – We first present two very general bounds following from Inequalities (1) and (3) and from the optimality of Markovian feedback discussed above and illustrated in Fig. 1 (see the supplemental material for the proofs). In the following, α_j^\uparrow will stand for the j -th smallest eigenvalue of the positive definite matrix $-(A + A^\top)$, while δ_j^\downarrow will stand for the j -th largest eigenvalue of the positive definite matrix D . We start by considering the achievable steady-state squeezing, defined as the smallest eigenvalue λ_1^\uparrow of the CM σ .

Proposition 1 (maximal squeezing). – Let σ be the CM of a steady-state achievable by continuous weak general-dyne measurements and linear driving in a system of bosonic modes subject to a drift matrix A and Gaussian white noise with a diffusion matrix D . The squeezing λ_1^\uparrow is bounded by

$$\lambda_1^\uparrow \geq \frac{\alpha_1^\uparrow}{\delta_1^\downarrow}. \quad (4)$$

Next, we turn our attention to the logarithmic negativity E_N , an entanglement monotone which provides one with an upper bound to the distillable entanglement, can be evaluated analytically for Gaussian states, and is a good quantifier of quantum correlations for systems which allow for no bound entanglement (like Gaussian bipartitions of 1 versus $(n-1)$ modes or bisymmetric [39] Gaussian bipartitions of any number of modes) [7, 41, 42].

Proposition 2 (maximal entanglement). – Let ϱ be the CM of a steady-state achievable by continuous weak general-dyne measurements and linear driving in a system of bosonic modes subject to a drift matrix A and Gaussian white noise with a diffusion matrix D . The logarithmic negativity $E_N(\varrho)$ of any 1 versus $(n-1)$ modes or bisymmetric bipartition of ϱ is bounded by

$$E_N(\varrho) \leq \max \left[0, \log_2 \left(\frac{\delta_1^\downarrow + \delta_2^\downarrow}{2\sqrt{\alpha_1^\uparrow \alpha_2^\uparrow}} \right) \right]. \quad (5)$$

A remarkable feature of both our bounds is that they increase (somewhat loosely, we will refer to λ_1^\uparrow getting smaller as an ‘increase’ in the squeezing) if the largest eigenvalues of the diffusion matrix D increase, which characterises a noisier environment. As mentioned above, if one considers a simple thermal environment, one has $\delta_1^\downarrow = \delta_2^\downarrow = 1 + 2N_1^\downarrow$, where N_1^\downarrow is the largest number of thermal excitations in an environmental degree of freedom. As we shall see, our bounds are actually tight in several important cases, where they represent the actual maximal values achievable. Hence, our findings show that the maximal achievable entanglement increases with the temperature of the bath.

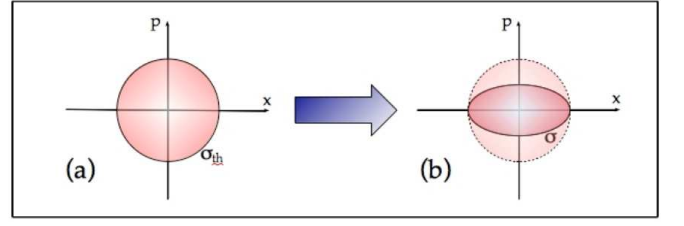


FIG. 2. Heuristic phase space picture of the noise-enhancement of the optimal feedback action. The feedback squashes the thermally broadened unconditional steady state CM σ_{th} , turning it into the squeezed CM σ . By Heisenberg principle, the squashing is limited by the inverse of the thermal uncertainty in the orthogonal quadrature, which increases with increasing noise.

This apparently counterintuitive property can be illustrated and understood by considering the feedback action on the squeezing of the *unconditional* state of a free single bosonic mode [43]. As we will see, in this case the optimal procedure to obtain squeezing consists in monitoring the environment along a given phase space direction and then in systematically driving the expectation value of the monitored quadrature to zero. As illustrated in Fig. 2, this produces an uncertainty contraction for that quadrature, while the conjugate, orthogonal quadrature is entirely unaffected. Hence, by Heisenberg principle, the achievable squeezing is ultimately limited by the inverse of the uncertainty in the orthogonal quadrature, which clearly increases with the available thermal energy of the bath. In a sense, this is a case of reservoir engineering where the effect of the bath is ‘squashed’ [44], rather than squeezed, by means of continuous measurements.

If one is interested in optimal squeezing, this thermal enhancement can be obtained by measuring and acting locally on a single quadrature, while the generation of optimal entanglement will generally require nonlocal measurements. It should however be noted here that linear feedback does allow for an increase in steady-state unconditioned entanglement even with local measurements, if the Hamiltonian couplings between the modes are strong enough [33]. Let us also mention that we have determined precise necessary and sufficient conditions for the bounds to be achievable (these are included in the supplemental material), and that, whenever the bound is achievable, the optimal steady-state is pure, because the saturation of the uncertainty relation (1) is implied. In such cases, the optimal feedback strategy not only maximises a figure of merit but also stabilises a pure state, regardless of how noisy the environment may be.

In what follows, we shall focus on specific examples and discuss findings of direct applicative interest. The reader may refer to the supplemental material for further technical details, such as explicit expressions for achievable optimal states and optimal unravellings.

Free systems. – Let us start with the simple case where no Hamiltonian is present ($H = 0$, which in practice corresponds to considering a system in the rotating frame, and to

having all the measurements' phase references rotate accordingly). Henceforth, we will always set $\delta_1^\downarrow = \delta_2^\downarrow = 1 + 2N_1^\downarrow$ (phase-insensitive thermal noise). Without any feedback action, the steady-state clearly corresponds to a thermal state without squeezing nor entanglement. On the other hand, the bounds on the squeezing and logarithmic negativity achievable via feedback read, respectively, $\lambda_1^\uparrow \geq 1/(1 + 2N_1^\downarrow)$ and $E_{\mathcal{N}} \leq \log_2(1 + 2N_1^\downarrow)$. The bound on the squeezing is always achievable by a simple feedback strategy: one has to perform a weak measurement on a certain (arbitrary) quadrature \hat{x}_ϕ interacting with the noisiest bath, and then drive the orthogonal quadrature with the corresponding current. The resulting state will be *optimally* squeezed in the measured quadrature \hat{x}_ϕ . A very instructive and consequential curiosity arises here: the efficiency of the optimal unravelling is not $\eta = 1$, but rather $\eta = N_1^\downarrow(1 + N_1^\downarrow)/[1 + N_1^\downarrow(1 + N_1^\downarrow)]$, which is always smaller than 1 for $N_1^\downarrow > 0$. This is a very explicit case of information-disturbance trade-off, whereby a weaker measurement (as is the case for lower efficiency) holds the virtue of causing less disturbance on the state, and might be desirable at times. This fact, besides its fundamental interest, might also have practical consequences, in systems where very few thermal excitations are the dominant source of noise, as could be the case at terahertz frequencies in solid-state and optical systems at room temperature [45, 46]: if $N_1^\downarrow \simeq 1$ (corresponding to about 30 THz), then the optimal detection efficiency is just $\eta \simeq 0.66$. It should however be noted that the optimal efficiency raises very quickly to 1, being already $\eta \simeq 0.86$ for $N_1^\downarrow \simeq 2$ (around 15 THz).

As for the entanglement, we can show that in the two-mode ($n = 2$) case, if the two baths have the same temperature N_1^\downarrow , the bound can be saturated. One of the possible optimal unravellings and Markovian strategies correspond to driving the quadratures $(\hat{x}_1 + \hat{x}_2)$ and $(\hat{p}_1 - \hat{p}_2)$ with the currents obtained by monitoring the conjugated quadratures $(\hat{p}_1 + \hat{p}_2)$ and $(\hat{x}_1 - \hat{x}_2)$, respectively. In this case an ideal homodyne measurement with unit efficiency is required to achieve the bound, and we hence considered explicitly also the case where this strategy is applied with inefficient measurements. Then, the logarithmic negativity achieved for efficiency η is given by $E_{\mathcal{N}} = \log_2(1 + 2N_1^\downarrow) - \log_2(1 + 4N_1^\downarrow(1 - \eta) + 4N_1^\downarrow(1 - \eta))$. By inspecting this equation one observes that, for a given temperature N_1^\downarrow , one can define a threshold value $\eta_{\text{th}} = \frac{1 + 2N_1^\downarrow}{2(1 + N_1^\downarrow)}$ such that entanglement is obtained only for efficiencies $\eta > \eta_{\text{th}}$. We notice that η_{th} is always greater than 1/2 and monotonically increases with temperature towards the maximum value corresponding to a perfect homodyne measurement.

In the most general case, when $N_1^\downarrow \neq N_2^\downarrow$, the bound cannot always be saturated. However, we were able to find an unravelling such that the steady state is a pure two-mode squeezed state with logarithmic negativity $E_{\mathcal{N}} = \log_2(1 + 2N_2^\downarrow)$.

Parametric Hamiltonians. – We now move on to consider

the case of degenerate parametric down conversion which can be described, in interaction picture, by the quadratic Hamiltonian $\hat{H} = \chi(\hat{x}_1\hat{p}_2 + \hat{p}_1\hat{x}_2)$ between two modes at the same frequency [47], such that the average number of thermal excitations in the two modes are the same and set equal to N : $D = (1 + 2N)\mathbb{1}_4$. We shall impose stability by bounding the interaction strength: $\chi < 1/2$. This set of dynamical parameters allows for the perfect saturation of the bound on the entanglement, and thus for the analytical optimisation of the achievable logarithmic negativity, which is given by $E_{\mathcal{N}} = \log_2(1 + 2N) - \log_2(1 - 2\chi)$, to be compared with the free steady state value $E_{\mathcal{N}}^{(0)} = \log_2(1 + 2\chi) - \log_2(1 + 2N)$ (what would be obtained in the absence of monitoring and feedback action). This is possibly the most apparent example of noise-enhanced performance in our study: while the free steady state logarithmic negativity decreases with the temperature, as one should expect, its optimised counterpart increases with N . The closed-loop control we considered is capable of retrieving information from the output channel and turning the phase insensitive thermal energy into correlations between the modes. The optimal feedback strategy has been exactly determined in this case as well, and corresponds, again, to driving the quadratures $(\hat{x}_1 + \hat{x}_2)$ and $(\hat{p}_1 - \hat{p}_2)$ with the currents obtained by monitoring the conjugated squeezed quadratures $(\hat{p}_1 + \hat{p}_2)$ and $(\hat{x}_1 - \hat{x}_2)$, respectively. In this case too a perfect homodyne measurement is required and one should hence consider the effect of the efficiency η on the achievable optimal entanglement. The conditions on the measurement efficiency for the feedback loop to be able to improve the generation of entanglement are rather strict, and become steeper and steeper as the noise increases. For $\chi = 0.3$ and $N = 1$, where the steady state in absence of feedback is unentangled, $\eta \geq 0.8$ is needed to generate any entanglement between the two set of modes. This threshold increases to 0.92 for $N = 2.5$ and to 0.98 for $N = 10$. As already mentioned above, these are hence the typical values of excitations where linear feedback control might really make a difference in the generation of pure entangled states of continuous variable systems.

Position couplings. – Let us also consider the other paradigmatic quadratic Hamiltonian $\hat{H} = \chi \hat{x}_1\hat{x}_2$ where, again, the two modes have the same frequency and the average number of thermal excitations in the two modes is set equal to N . This is the typical form of linearised interaction involving material harmonic oscillators, such as Coulomb-coupled trapped ions or mirrors coupled to cavity field modes in optomechanical set-ups. In this case, one observes that the bound obtained by means of our theorem is not achievable, since the conditions outlined in the supplemental material are not satisfied. However, we were able to identify a family of two-mode squeezed states which are stabilising solution of the conditional dynamics and whose logarithmic negativity reads $E_{\mathcal{N}} = \log_2(2(1 + 2N)) - \log_2(\chi + 2)$. As is apparent from the formula, the entanglement of these states increases with the thermal excitations number N , which proves that a thermal environment can boost the entanglement of $\hat{x}\hat{x}$ coupled

oscillators as well.

Conclusions. – In this Letter, we have shown that linear feedback may be employed not only to stave off the effect of thermal noise and achieve pure steady states [48, 49], but also to convert the thermal energy of a bath into enhanced squeezed or entangled resources, of paramount importance to precision measurements [50–52], quantum information processing [53, 54], and quantum communication with continuous variable quantum systems [55, 56].

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* m.genoni@imperial.ac.uk

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SUPPLEMENTAL MATERIAL

LINEAR QUANTUM SYSTEMS AND OPTIMAL UNRAVELLINGS

In this section we recall the description of the evolution under continuous measurements for linear quantum system and we add some additional information regarding their optimal unravelling.

We start by dealing with time-independent quadratic Hamiltonians

$$\hat{H} = (1/2)\hat{\mathbf{R}}^T H \hat{\mathbf{R}}, \quad \hat{\mathbf{R}} = (\hat{x}_1, \hat{p}_1, \dots, \hat{x}_n, \hat{p}_n)^T.$$

As usual, the system is considered to be open and linearly coupled to a large uncorrelated Markovian bath. As explained in the paper, all the dynamics is defined by the ‘drift matrix’ A and by the diffusion matrix D , such that, in terms of the covariance matrix of the system, we have

$$\frac{d\sigma}{dt} = A\sigma + \sigma A^T + D. \quad (6)$$

Given these matrices, and the Hamiltonian \hat{H} , we can define the matrix

$$M = \frac{\Omega D \Omega^T}{2} - i(H + \Omega A), \quad (7)$$

and one can show, given the properties of A , D , and H , that this matrix is always hermitian and positive. Thus it can be diagonalized as $M = V\Lambda V^\dagger$, where V is unitary and Λ is a diagonal matrix and we can define the matrix $\tilde{C} = \Lambda^{1/2}V^\dagger$, and the vector of operators $\hat{\mathbf{c}} = \tilde{C}\hat{\mathbf{R}}$. For instance, by considering the two-mode case ($n = 2$) and $D = (2N + 1)\mathbb{1}_4$, the matrix \tilde{C} and the vector $\hat{\mathbf{c}}$ read

$$\tilde{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{N_1+1} & i\sqrt{N_1+1} & 0 & 0 \\ \sqrt{N_1} & -i\sqrt{N_1} & 0 & 0 \\ 0 & 0 & \sqrt{N_2+1} & i\sqrt{N_2+1} \\ 0 & 0 & \sqrt{N_2} & -i\sqrt{N_2} \end{pmatrix}, \quad (8)$$

$$\hat{\mathbf{c}} = \{a_1\sqrt{N+1}, a_1^\dagger\sqrt{N}, a_2\sqrt{N+1}, a_2^\dagger\sqrt{N}\}^T. \quad (9)$$

We now assume that the degrees of freedom of the environment can be continuously monitored, resulting in instantaneous Gaussian measurements on the system. These measurements, which are referred to as general-dyne [1], can be parametrized by an unravelling matrix U ,

$$U = \frac{1}{2} \begin{pmatrix} \Theta + \text{Re}[\Upsilon] & \text{Im}[\Upsilon] \\ \text{Im}[\Upsilon] & \Theta - \text{Re}[\Upsilon] \end{pmatrix} \quad (10)$$

where Θ and Υ are square matrices typically of dimension $2n \times 2n$. To be a valid measurement, U has to be positive semi-definite, $U \geq 0$, and has to satisfy the following other constraints: $\Upsilon = \Upsilon^T$ and $\Theta = \text{diag}(\eta_1, \dots, \eta_L)$, where

$\eta_l \in [0, 1]$. The matrix U (or equivalently the pair of matrices Θ and Υ) corresponds physically to weak non-demolition measurements of the operators $(\hat{\mathbf{c}}^T \Theta + \hat{\mathbf{c}}^\dagger \Upsilon)$, which comprises all Gaussian measurements, including homodyne and heterodyne detection. In particular, the values η_l characterizing the matrix Θ , correspond to the quantum efficiencies of the measurements on each environment mode. In our study we have always considered a common fixed efficiency $\eta_l = \eta$ for each mode of the bath.

Under such measurements, we obtain for the conditional state a diffusive equation with a stochastic component for the first moments $\langle \hat{\mathbf{R}} \rangle_c$ and a deterministic equation for the CM σ_c , in formulae:

$$d\langle \hat{\mathbf{R}} \rangle_c = A\langle \hat{\mathbf{R}} \rangle_c dt + (\sigma_c C^T + \Gamma^T) d\mathbf{w} \quad (11)$$

$$\frac{d\sigma_c}{dt} = A\sigma_c + \sigma_c A^T + D - (\sigma_c C^T + \Gamma^T)(C\sigma_c + \Gamma) \quad (12)$$

where $d\mathbf{w}$ is a vector of real Wiener increments satisfying $d\mathbf{w}d\mathbf{w}^T = \mathbb{1}_{2n} dt$ [2], $\Gamma = (2U)^{1/2} S \bar{C} \Omega$, $\bar{C}^T = (\text{Re}[\tilde{C}^T], \text{Im}[\tilde{C}^T])$ and

$$S = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}.$$

Then a linear time-dependent term is added to the Hamiltonian

$$\hat{H}_f = -\hat{\mathbf{R}}^T \Omega B \mathbf{y}(t), \quad (13)$$

where $\mathbf{y}(t)$ is the current obtained from the continuous measurement and the matrix B defines the kind of Markovian feedback action exerted on the system. In this case, the evolution equation for the unconditional state covariance matrix σ is still of the form $(d\sigma)/dt = A'\sigma + \sigma A'^T + D'$, where

$$A' = A + BC, \quad (14)$$

$$D' = D - C^T B^T - BC + 2BB^T, \quad (15)$$

and $C = 2(U)^{1/2} \bar{C}$.

A CM σ_c is a physical stabilising solution of the conditional dynamics if it satisfies the two following conditions (see Eq. (12) and notice that the second term on the right-hand side is always positive)

$$\sigma_c + i\Omega \geq 0 \quad (\text{physicality condition}), \quad (16)$$

$$A\sigma_c + \sigma_c A^T + D \geq 0 \quad (\text{stabilising condition}). \quad (17)$$

As derived by Wiseman and Doherty [3], given a stabilising CM σ_c , an optimal unravelling U_{opt} such that σ_c can be obtained at steady-state, always exists. In particular a (not necessarily unique) optimal unravelling U_{opt} can be obtained by solving the equation

$$2E^T U E = D + A\sigma_c + \sigma_c A^T, \quad (18)$$

where $E = \bar{C}\sigma_c + S\bar{C}\Omega$. The Hamiltonian term in Eq. (13) is then chosen so as to cancel out the first moments and make

the average unconditional state coincide with the conditional state. It can be shown that the matrix B_{opt} achieving this, for a given steady-state CM σ_c , reads

$$B_{\text{opt}} = -\sigma_c C^T - \Gamma^T. \quad (19)$$

BOUNDS ON MAXIMUM ACHIEVABLE SQUEEZING AND ENTANGLEMENT

In the following we derive the bound on the Gaussian entanglement and squeezing achievable by means of any feedback strategy where the control action consists in linear driving. We present our main findings as three lemmas leading to two final propositions.

Lemma 1 (Bound on smallest symplectic eigenvalue) *The smallest partially transposed symplectic eigenvalue $\tilde{\nu}_-$ of a generic CM σ is bounded from below as follows*

$$\tilde{\nu}_-^2 \geq \lambda_1^\uparrow \lambda_2^\uparrow, \quad (20)$$

λ_1^\uparrow and λ_2^\uparrow being the two smallest eigenvalues of σ .

Proof Notice that this proof can be found in [4]. We will reproduce it here to make our work self-contained.

Henceforth, $|v\rangle$ will stand for a unit vector in the phase space Γ and $\langle v|$ will be its dual under the Euclidean scalar product. Also, given a bipartition of the modes into the ‘first’ l and the ‘last’ m modes, let us define the matrix T , representing partial transposition in phase-space, as $T = \mathbb{1}_2^{\oplus l} \oplus \sigma_z^{\oplus m}$, σ_z being the z Pauli matrix. Hence, the partially transposed symplectic form is defined as $\tilde{\Omega} = T\Omega T$

The squared symplectic eigenvalue $\tilde{\nu}_-^2$ is the smallest eigenvalue of the matrix $\sigma^{1/2} \tilde{\Omega}^T \sigma \tilde{\Omega} \sigma^{1/2}$:

$$\tilde{\nu}_-^2 = \min_{|v\rangle} \langle v | \sigma^{1/2} \tilde{\Omega}^T \sigma \tilde{\Omega} \sigma^{1/2} | v \rangle.$$

For each $|v\rangle$, one can define the unit vector $|w\rangle = \tilde{\Omega} \sigma^{1/2} |v\rangle / \sqrt{\langle v | \sigma | v \rangle}$, such that $\langle v | \sigma^{1/2} | w \rangle = 0$ (due to the antisymmetry of $\tilde{\Omega}$) and

$$\tilde{\nu}_-^2 = \min_{|v\rangle} \langle v | \sigma | v \rangle \langle w | \sigma | w \rangle \geq \min_{|v\rangle, |w\rangle} \langle v | \sigma | v \rangle \langle w | \sigma | w \rangle = \lambda_1^\uparrow \lambda_2^\uparrow.$$

The last equality is easily verified once $\langle v | \sigma^{1/2} | w \rangle = 0$ and $\sigma > 0$ are enforced, and completes the proof. \square

Next, the uncertainty principle entails:

Lemma 2 (Uncertainty relation for CMs’ eigenvalues) *Let $\{\lambda_j^\uparrow\}$ and $\{\lambda_j^\downarrow\}$ be, respectively, the $2n$ increasingly-ordered and decreasingly-ordered eigenvalues of an n -mode CM σ . Then one has:*

$$\lambda_j^\uparrow \lambda_j^\downarrow \geq 1 \quad \text{for } 1 \leq j \leq n. \quad (21)$$

Proof Note that the uncertainty relation (16) is equivalent to the two following conditions [4, 5]:

$$\sigma^{1/2} \Omega^T \sigma \Omega \sigma^{1/2} \geq 1, \quad \text{and} \quad \sigma > 0. \quad (22)$$

For any $|v\rangle \in \Gamma$ one can define $|z\rangle = \Omega \sigma^{1/2} |v\rangle / \sqrt{\langle v | \sigma | v \rangle}$, so that the Robertson Schrödinger Inequality (22) can be recast as

$$\langle v | \sigma | v \rangle \langle z | \sigma | z \rangle \geq 1 \quad \forall |v\rangle \in \Gamma. \quad (23)$$

We will now denote by $|v_j\rangle$ the eigenvectors corresponding to the increasingly ordered eigenvalues of σ : $\sigma |v_j\rangle = \lambda_j^\uparrow |v_j\rangle$. Let us consider a vector $|v\rangle$ belonging to the subspace, which we shall denote Γ_k , spanned by the k smallest eigenvectors of σ $\{|v_j\rangle\}$, for $j \leq k$. Clearly one has $\langle v | \sigma | v \rangle \leq \lambda_k^\uparrow$. The inequality (23) then leads to

$$\lambda_k^\uparrow \langle z | \sigma | z \rangle \geq \langle v | \sigma | v \rangle \langle z | \sigma | z \rangle \geq 1 \quad \forall |v\rangle \in \Gamma_k,$$

which must be satisfied by all the vectors $|z\rangle$ belonging to the k -dimensional linear subspace $\Omega \Gamma_k$ (defined as the subspace spanned by the k orthogonal vectors $\Omega |v_k\rangle$):

$$\lambda_k^\uparrow \langle z | \sigma | z \rangle \geq 1 \quad \forall |z\rangle \in \Omega \Gamma_k.$$

By Poincaré Inequality [6], a vector $|z\rangle$ must exist in $\Omega \Gamma_k$ for which $\langle z | \sigma | z \rangle \leq \lambda_k^\downarrow$, such that $\lambda_k^\uparrow \lambda_k^\downarrow \geq 1$. \square

As an immediate corollary of Lemma 2, one obtains

$$\lambda_1^\uparrow \lambda_2^\uparrow \geq \frac{1}{\lambda_1^\downarrow \lambda_2^\downarrow}. \quad (24)$$

Lemma 3 (Bound on eigenvalues of steady state CMs)

Let σ_c be a conditional CM at steady state obtained under continuous Gaussian measurements, diffusion matrix D and a Hamiltonian matrix H . The product of the two largest eigenvalues λ_1^\downarrow and λ_2^\downarrow of σ_c is bounded as follows:

$$\lambda_1^\downarrow \lambda_2^\downarrow \leq \frac{(\delta_1^\downarrow + \delta_2^\downarrow)^2}{4 \alpha_1^\uparrow \alpha_2^\uparrow} \quad (25)$$

where $\{\alpha_j^\uparrow\}$ are the (strictly positive) eigenvalues of $(-A - A^T)$ in increasing order, while $\{\delta_j^\downarrow\}$ are the (strictly positive) eigenvalues of D in decreasing order.

Proof Given the condition (17), and given the eigenvectors of σ_c , $|\lambda_1^\downarrow\rangle$ and $|\lambda_2^\downarrow\rangle$ corresponding to λ_1^\downarrow and λ_2^\downarrow , we have

$$\lambda_1^\downarrow \langle \lambda_1^\downarrow | - (A + A^T) | \lambda_1^\downarrow \rangle \leq \langle \lambda_1^\downarrow | D | \lambda_1^\downarrow \rangle \quad (26)$$

$$\lambda_2^\downarrow \langle \lambda_2^\downarrow | - (A + A^T) | \lambda_2^\downarrow \rangle \leq \langle \lambda_2^\downarrow | D | \lambda_2^\downarrow \rangle \quad (27)$$

By defining $\tilde{A} = -(A + A^T)$ and multiplying the inequalities, we have

$$\lambda_1^\downarrow \lambda_2^\downarrow \langle \lambda_1^\downarrow | \tilde{A} | \lambda_1^\downarrow \rangle \langle \lambda_2^\downarrow | \tilde{A} | \lambda_2^\downarrow \rangle \leq \langle \lambda_1^\downarrow | D | \lambda_1^\downarrow \rangle \langle \lambda_2^\downarrow | D | \lambda_2^\downarrow \rangle \quad (28)$$

then

$$\lambda_1^\downarrow \lambda_2^\downarrow \leq \frac{\max_{\langle v_1|v_2 \rangle=0} \langle v_1|D|v_1 \rangle \langle v_2|D|v_2 \rangle}{\min_{\langle v_1|v_2 \rangle=0} \langle v_1|\tilde{A}|v_1 \rangle \langle v_2|\tilde{A}|v_2 \rangle} \quad (29)$$

$$\leq \frac{(\delta_1^\downarrow + \delta_2^\downarrow)^2}{4 \alpha_1^\uparrow \alpha_2^\uparrow} \quad (30)$$

where we use

$$\min_{\langle v_1|v_2 \rangle=0} \langle v_1|\tilde{A}|v_1 \rangle \langle v_2|\tilde{A}|v_2 \rangle \geq \alpha_1^\uparrow \alpha_2^\uparrow \quad (31)$$

$$\max_{\langle v_1|v_2 \rangle=0} \langle v_1|D|v_1 \rangle \langle v_2|D|v_2 \rangle \leq \left(\frac{\delta_1^\downarrow + \delta_2^\downarrow}{2} \right)^2 \quad (32)$$

□

Further, and more generally, one has:

Proposition 1 (Maximal unconditional squeezing) *Let σ be the CM of a steady-state achievable by continuous weak general-dyne measurements and linear driving in a system of bosonic modes subject to a drift matrix A and Gaussian white noise with a diffusion matrix D . The squeezing λ_1^\downarrow is bounded by*

$$\lambda_1^\downarrow \geq \frac{\alpha_1^\uparrow}{\delta_1^\downarrow}. \quad (33)$$

Proof From Eq. (21) we obtain the relation $\lambda_1^\downarrow \geq 1/\lambda_1^\uparrow$, where λ_1^\uparrow (λ_1^\downarrow) is the largest (smallest) eigenvalue of a CM σ . By considering a conditional CM at steady state and following the same line of reasoning used in Lemma 3, we obtain the following inequality $\lambda_1^\downarrow \leq \delta_1^\downarrow/\alpha_1^\uparrow$, which yields the inequality:

$$\lambda_1^\downarrow \geq \frac{1}{\lambda_1^\uparrow} \geq \frac{\alpha_1^\uparrow}{\delta_1^\downarrow}. \quad (34)$$

As explained before, the unconditional state ϱ that we obtain from our dynamics is a statistical mixture (with Gaussian profile) of different conditional states $\varrho_{\mathbf{r}}$ having the same CM σ_c and different first moments $\mathbf{r} = \langle \hat{\mathbf{R}} \rangle_c$, in formulae $\varrho = \int d\mathbf{r} p(\mathbf{r}) \varrho_{\mathbf{r}}$. As a consequence, the unconditional CM reads $\sigma = \sigma_c + \tau$ where $\tau > 0$ is the classical covariance matrix of the first moments' distribution $p(\mathbf{r})$. Thus the lowest eigenvalue of σ is lower bounded by the eigenvalue of σ_c and the bound above is valid for the unconditional state. It is worth to remember that, given an optimal CM σ_c which is a physical stabilising solution of the conditional dynamics, the bound is tight, since we can always find a Markovian feedback strategy such that $\varrho = \varrho_{\mathbf{r}=0}$, that is such that the unconditional state has CM σ_c and zero first moments. □

Proposition 2 (Maximal unconditional entanglement) *Let ϱ be the CM of a steady-state achievable by continuous weak general-dyne measurements and linear driving in a system of bosonic modes subject to a drift matrix A and Gaussian white noise with a diffusion matrix D . The logarithmic negativity*

$E_{\mathcal{N}}(\varrho)$ [7] of any 1 versus $(n-1)$ modes or bisymmetric bipartition of ϱ is bounded by

$$E_{\mathcal{N}}(\varrho) \leq \max \left[0, \log_2 \left(\frac{\delta_1^\downarrow + \delta_2^\downarrow}{2\sqrt{\alpha_1^\uparrow \alpha_2^\uparrow}} \right) \right]. \quad (35)$$

Proof The chain of Inequalities (20), (24) and (25) leads to

$$\tilde{\nu}_-^2 \geq \frac{4 \alpha_1^\uparrow \alpha_2^\uparrow}{(\delta_1^\downarrow + \delta_2^\downarrow)^2}, \quad (36)$$

which, in turn, constrains the maximal logarithmic negativity achievable for states $\varrho_{\mathbf{r}}$ conditioned by Gaussian measurements having a CM σ_c . In fact, by using the formula $E_{\mathcal{N}} = \max[0, -\log(\tilde{\nu}_-)]$, we obtain,

$$E_{\mathcal{N}}(\varrho_{\mathbf{r}}) \leq \max \left[0, \log_2 \left(\frac{\delta_1^\downarrow + \delta_2^\downarrow}{2\sqrt{\alpha_1^\uparrow \alpha_2^\uparrow}} \right) \right]. \quad (37)$$

On the other hand the unconditional (Gaussian) state reads $\varrho = \int d\mathbf{r} p(\mathbf{r}) \varrho_{\mathbf{r}}$; this implies that ϱ can be obtained from the Gaussian state $\varrho_{\mathbf{r}=0}$ (having CM σ_c and vanishing first moments) by local operations and classical communication alone, because first moments can be arbitrarily adjusted by local unitary operations. Since the log-negativity is an entanglement monotone [8], we have $E_{\mathcal{N}}(\varrho) \leq E_{\mathcal{N}}(\varrho_{\mathbf{r}})$, that is the bound above is valid also for the unconditional state and can be achieved by means of optimal Markovian feedback. □

Necessary conditions for the tightness of the bounds

By considering how our bounds were derived, and working backward, we can determine sharp conditions on the matrices A and D , which fully characterise our dynamics, for the bounds to be achievable. In order to express such conditions, let us define the eigenvectors $|\alpha_j^\uparrow\rangle$ and $|\delta_j^\downarrow\rangle$ associated, respectively, to the j -th smallest eigenvalue of $\tilde{A} = -A - A^\top$ and j -th largest eigenvalue of D . This leads to the following two additional propositions:

Proposition 3 (Conditions for maximal squeezing) *A continuously measured and linearly driven Gaussian system is able to saturate the bound (33) if and only if*

$$|\alpha_1^\uparrow\rangle = |\delta_1^\downarrow\rangle. \quad (38)$$

Proof The inequality $\lambda_1^\downarrow \leq \delta_1^\downarrow/\alpha_1^\uparrow$ (the analogous of (30) for the squeezing case) is only saturated if the eigenvector associated to the largest eigenvalue of σ coincides with $|\alpha_1^\uparrow\rangle$ and $|\delta_1^\downarrow\rangle$, hence our condition (38), in that it is always possible to construct a physical σ with largest eigenvalue along a particular direction. □

Proposition 4 (Conditions for maximal entanglement) *A continuously measured and linearly driven Gaussian system is able to saturate the bound (37) if and only if the following relationships are satisfied*

$$|\alpha_1^\uparrow\rangle = \frac{|\delta_1^\downarrow\rangle \mp |\delta_2^\downarrow\rangle}{\sqrt{2}}, \quad (39)$$

$$|\alpha_2^\uparrow\rangle = \frac{|\delta_1^\downarrow\rangle \pm |\delta_2^\downarrow\rangle}{\sqrt{2}}, \quad (40)$$

$$|\alpha_2^\uparrow\rangle = \Omega^\top \tilde{\Omega} \Omega |\alpha_1^\uparrow\rangle, \quad (41)$$

$$\langle \alpha_1^\uparrow | T | \alpha_1^\uparrow \rangle = 0 \quad (42)$$

(where \mp and \pm mean that if Eq. (39) has a minus sign then Eq. (40) has a plus, and viceversa, and that either choice is a valid condition).

Proof Eqs. (39) and (40) are necessary for the saturation of Inequality (30), along with the choices $|\lambda_1^\downarrow\rangle = |\alpha_1^\uparrow\rangle$ and $|\lambda_2^\downarrow\rangle = |\alpha_2^\uparrow\rangle$. Then, inspection of the proof of Lemma 2 for $k = 1, k = 2$ and, by induction, for any k , reveals that the condition $\lambda_k^\downarrow \lambda_k^\uparrow = 1$ is saturated if and only if $|\lambda_k^\uparrow\rangle = \Omega |\lambda_k^\downarrow\rangle$ (where the eigenvectors associated to λ_k^\downarrow and λ_k^\uparrow have been denoted with $|\lambda_k^\downarrow\rangle$ and $|\lambda_k^\uparrow\rangle$). Now, in order to saturate the bound, this additional condition can only be imposed if the two eigenvectors $|\lambda_k^\downarrow\rangle$, already determined by (39) and (40), are orthogonal to $\Omega |\lambda_k^\downarrow\rangle$ (so that the latter can also be eigenvectors of σ), that is

$$\langle \lambda_2^\downarrow | \Omega | \lambda_1^\downarrow \rangle = \langle \alpha_2^\downarrow | \Omega | \alpha_1^\downarrow \rangle = 0. \quad (43)$$

Further, inspection of Lemma 1 shows that, for Inequality (20) to be saturated, it must be $|\lambda_2^\uparrow\rangle = \tilde{\Omega} |\lambda_1^\uparrow\rangle$ which, by the conditions $|\lambda_1^\downarrow\rangle = |\alpha_1^\uparrow\rangle$, $|\lambda_2^\downarrow\rangle = |\alpha_2^\uparrow\rangle$ and $|\lambda_k^\uparrow\rangle = \Omega |\lambda_k^\downarrow\rangle$ imposed at previous steps, becomes $|\alpha_2^\uparrow\rangle = \Omega^\top \tilde{\Omega} \Omega |\alpha_1^\uparrow\rangle$, which proves condition (41). Finally, by inserting Eq. (41) into Eq. (43), and noting that $\Omega^\top \tilde{\Omega} \Omega \Omega = -T$, one can recast condition (43) in terms of $|\alpha_1^\uparrow\rangle$ alone as condition (42). \square

APPLICATIONS.

OPTIMAL ACHIEVABLE STATES AND UNRAVELLINGS

In this section we give some details regarding the optimal states and the optimal unravellings in the different cases we discuss in the paper.

Free systems. - Let us start by considering the maximum single-mode squeezing achievable when no Hamiltonian is present. For a given number of thermal excitations N , one can achieve a squeezed vacuum state $|\psi\rangle = S(r_{\text{opt}} e^{i\phi})|0\rangle$, where $S(\xi) = \exp\{\frac{1}{2}\xi(a^\dagger)^2 - \frac{1}{2}\xi^* a^2\}$ is the single-mode squeezing operator and

$$r_{\text{opt}} = \frac{1}{2} \ln \left(\frac{1}{1+2N} \right). \quad (44)$$

Since there is no privileged direction in the phase-space, one can choose the phase of the squeezing ϕ by choosing one of the optimal measurements on the environment. In fact the optimal unravelling and feedback strategy correspond to perform a homodyne measurement on a quadrature $\hat{x}_\phi = \cos \phi \hat{x} + \sin \phi \hat{p}$, and then to drive the orthogonal quadrature with the obtained current. For instance, if one measures \hat{x} , the optimal unravelling matrix reads

$$U_{\text{opt}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{N(1+N)}{1+N+N^2} & 0 \\ 0 & 0 & 0 & \frac{N(1+N)}{1+N+N^2} \end{pmatrix} \quad (45)$$

and the steady-state will be squeezed in the \hat{x} quadrature. The two matrices that parametrize the measurement, are

$$\Theta_{\text{opt}} = -\Upsilon_{\text{opt}} = \eta_{\text{opt}} \mathbb{I}_2, \quad (46)$$

where

$$\eta_{\text{opt}} = \frac{N(1+N)}{1+N+N^2}. \quad (47)$$

As explained in the paper, this corresponds to an inefficient homodyne measurement, with efficiency η_{opt} .

As regards the entanglement, we focus here on the two-mode case where the two baths have the same number of thermal excitations N . We find that a two-mode squeezed state $|\psi\rangle = S_2(r_{\text{opt}} e^{i\phi})|0\rangle$, is a stabilising solution of the dynamics and achieves maximum value of entanglement given by the bound we have derived. We denote with $S_2(\xi) = \exp\{\xi a^\dagger b^\dagger - \xi^* ab\}$ the two-mode squeezing operator, while the squeezing parameter r_{opt} is still equal to the one given in (44). As in the previous case, there is no privileged direction in the phase-space and thus one can consider different optimal unravellings, giving different squeezing phases ϕ . One of them corresponds to the measure of the quadratures $\hat{x}_1 - \hat{x}_2$ and $\hat{p}_1 + \hat{p}_2$, which in terms of the unravelling matrix reads

$$U_{\text{opt}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}. \quad (48)$$

By looking at the corresponding matrices

$$\Upsilon_{\text{opt}} = \begin{pmatrix} 0 & -\mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix} \quad (49)$$

$$\Theta = \mathbb{I}_4 \quad (50)$$

one observes that in this case the optimal unravelling requires a perfect homodyne detection on the non-local quadratures. In the results we present in the paper, we also consider the role of

the efficiency on the achievable entanglement; this can be investigated by just replacing the optimal unravelling, with the matrix $U = \eta U_{\text{opt}}$, where $\eta \in [0, 1]$ denotes the homodyne efficiency which is considered to be equal for both modes involved.

Parametric Hamiltonians. - We consider here the case of degenerate parametric down conversion, described by the quadratic Hamiltonian

$$\hat{H} = \chi(\hat{x}_1\hat{p}_2 + \hat{p}_1\hat{x}_2)$$

between two modes at the same frequency, such that the average number of thermal excitations in the two-modes are the same and set equal to N . In this case the coupling defines a privileged direction in the phase space, since it squeezes the quadratures $\hat{x}_1 - \hat{x}_2$ and $\hat{p}_1 + \hat{p}_2$. Indeed, the two-mode squeezed state which is stabilising solution of the dynamics and achieves the bound on the entanglement, reads $S_2(\tilde{r}_{\text{opt}})|0\rangle$, where

$$\tilde{r}_{\text{opt}} = \frac{1}{2} \ln \left(\frac{1 + 2N}{1 - 2\chi} \right). \quad (51)$$

The state is squeezed, as expected, in the two non-local

quadratures indicated above. Also the optimal unravelling that we obtain with Eq. (18) reflects this fact. It corresponds to the matrix given in Eq. (48), and thus to a perfect homodyne measurement of the quadratures $\hat{x}_1 - \hat{x}_2$ and $\hat{p}_1 + \hat{p}_2$. In this case too, results for inefficient measurements, that is by considering the unravelling $U = \eta U_{\text{opt}}$, are reported in the Letter.

* m.genoni@imperial.ac.uk

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